

European and Asian Greeks for exponential Lévy processes

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Abstract

In this article we use Malliavin Calculus in order to find closed expressions for European and Asian Greeks where the underlying asset is driven by an exponential Lévy process. This approach allows to extend the Monte Carlo Malliavin method to the Greeks of exponential jump diffusions.

1 Introduction

One of the classical applications of Malliavin Calculus is to find closed expressions of the so-called Greeks which are partial derivatives of the estimated option prices with respect to certain parameters like the initial price, the volatility and the risk-free interest rate. Let X denote the underlying financial asset, let Φ be the payoff function of the underlying option, let T be the exercise time of the option and let r denote the risk-free interest rate. Then the value V of the option (or expected outcome) can be expressed as

$$V_E = E[e^{-rT}\Phi(X_T)] \quad (1)$$

for a European option and

$$V_A = E\left[e^{-rT}\Phi\left(\int_0^T X_t dt\right)\right] \quad (2)$$

for an Asian option. This paper will be concerned with put- and call options. The payoff function of a call option is

$$\Phi_c(x) = \max\{0, (x - K)\} = (x - K)^+, \quad (3)$$

and the payoff function of a put option is

$$\Phi_p(x) = \max\{0, (K - x)\} = (K - x)^+, \quad (4)$$

where K is the **exercise- or strike price**.

Let $\Delta = \frac{\partial}{\partial x}V$ be the Greek Δ , i.e. the derivative of the option value with respect to the initial

value x of the underlying process X . So defined Δ can be expressed as

$$\Delta = \frac{\partial}{\partial x} E \left[e^{-rT} \Phi \left(\int_0^T X_t dt \right) \right] \quad (5)$$

for an Asian option. A straightforward method for the numerical approximation of Greeks is the Monte Carlo finite difference method. An alternative method is the Monte Carlo Malliavin Method. Malliavin Calculus is used to find a stochastic weight π s.t. the derivative of the option value can be expressed as

$$\Delta = E \left[e^{-rT} \Phi \left(\int_0^T X_t dt \right) \pi \right]. \quad (6)$$

The value of Δ is then computed by the Monte Carlo method. This method was first introduced in the article [3] where Malliavin Calculus is applied to derive closed formulas for Greeks in the Black Scholes model. In this model, the price process is given by

$$dX_t = rX_t dt + \sigma X_t dW_t; \quad X_0 = x, \quad (7)$$

where $r \in \mathbb{R}$ is the risk-free interest rate, $\sigma \in \mathbb{R}_{>0}$ the volatility and $x \in \mathbb{R}_{>0}$ is the initial condition. There are many approaches to generalize the formulas of European Greeks to more general jump diffusions, see for example [1] and [2]. Our approach generalizes these results to even more general jump diffusions and it also allows us to derive closed formulas for Asian Greeks.

In the first part of this article we will briefly introduce to the Hilbert space-valued Malliavin Calculus as given the treatment in [5]. Then we apply this method to the calculation of European and Asian Greeks for general jump diffusion models.

2 Setting and Notation

Let $(\tilde{X}_t)_{t \in [0, T]}$, $\tilde{X}_0 = x$ be a Lévy process with nonvanishing Brownian motion part on a probability space (Ω, \mathcal{F}, P) such that $X_t \in L^2(\Omega)$ for all $t \in [0, T]$. Then there is a geometric Brownian motion $X_t^{(1)} = \gamma t + \sigma W_t = (r - \sigma^2/2 + \tilde{\gamma})t + \sigma W_t$, $\sigma \neq 0$ on a probability space $(\Omega_W, \mathcal{F}_W, P_W)$ and a pure jump process X^J on a probability space $(\Omega_J, \mathcal{F}_J, P_J)$ s.t.

$$(\Omega, \mathcal{F}, P) \simeq (\Omega_W, \mathcal{F}_W, P_W) \otimes (\Omega_J, \mathcal{F}_J, P_J) \quad (8)$$

and s.t. under this identification

$$\tilde{X}_t = \gamma t + \sigma W_t + X_t^J \quad (9)$$

holds. We assume, that W is a standard Brownian motion. The process X^J can furthermore be decomposed into the sum of a compound Poisson process $X^{(2)} = \sum_{i=1}^{N_t} \alpha Y_i$ and a square integrable pure jump martingale $X^{(3)}$ that almost surely has a countable number of jumps on finite intervals,

i.e.

$$X_t^J = \sum_{i=1}^{N_t} \alpha Y_i + X_t^{(3)}. \quad (10)$$

Let λ denote the intensity of the Poisson process N . By means of the natural identification $L^2(\Omega_W \times \Omega_J) \simeq L^2(\Omega_W, L^2(\Omega_J))$ we can regard every random variable $Y \in L^2(\Omega)$ as a random variable on Ω_W with values in the Hilbert space $L^2(\Omega_J)$.

3 The Malliavin Calculus for $L^2(\Omega_J)$ -valued Processes

This approach is based on the usual Malliavin Calculus in the Brownian motion case as treated in [4]. The following short introduction to this topic is based on [5].

We write $\mathbb{D}^{1,2} \subseteq L^2(\Omega_W)$ for the domain of the Malliavin derivative D and write δ for its adjoint operator. The following definition extends the notion of Malliavin derivative to the Hilbert space case.

Definition 3.1. For $X = \sum_{i=1}^n F_i v_i \in \mathbb{D}^{1,2} \otimes L^2(\Omega_J)$, i.e. $F_i \in \mathbb{D}^{1,2}$ and $v_i \in L^2(\Omega_J)$ for all $i = 1, \dots, n$ define the Malliavin derivative D by

$$DX := \sum_{i=1}^n DF_i \otimes v_i. \quad (11)$$

D is a closable operator with domain in $L^2(\Omega_W, L^2(\Omega_J))$ and values in $L^2([0, T] \times \Omega_W, L^2(\Omega_J))$. We denote the closure of its domain by $\mathbb{D}^{1,2}(L^2(\Omega_J))$.

We use the same symbol D for the derivative operator on $\mathbb{D}^{1,2}$ and on $\mathbb{D}^{1,2}(L^2(\Omega_J))$, but it becomes clear from the context which operator is meant.

The product of two random variables $X = \sum_{i=1}^n F_i v_i \in \mathbb{D}^{1,2} \otimes L^2(\Omega_J)$ and $Y = \sum_{j=1}^m G_j w_j \in \mathbb{D}^{1,2} \otimes L^2(\Omega_J)$ is defined pointwise, i.e.

$$XY(\omega)(\omega') := \sum_{1 \leq i \leq n, 1 \leq j \leq m} F_i(\omega) G_j(\omega) v_i(\omega') w_j(\omega'). \quad (12)$$

This definition extends naturally to the whole domain of $\mathbb{D}^{1,2}(L^2(\Omega_J))$.

Proposition 3.2. *The Malliavin derivative on $\mathbb{D}^{1,2}(L^2(\Omega_J))$ satisfies the product rule: For $X, Y \in \mathbb{D}^{1,2}(L^2(\Omega_J))$ we have $XY \in \mathbb{D}^{1,2}(L^2(\Omega_J))$ and*

$$D(XY) = XDY + YDX. \quad (13)$$

Proof. It suffices to show this property on $\mathbb{D}^{1,2} \otimes L^2(\Omega_J)$. Let $X, Y \in \mathbb{D}^{1,2} \otimes L^2(\Omega_J)$ have the representations $X = \sum_{i=1}^n F_i v_i$ and $Y = \sum_{j=1}^m G_j w_j$ with $F_i, G_j \in \mathbb{D}^{1,2}$ and $v_i, w_j \in L^2(\Omega_J)$ for all $i = 1, \dots, n, j = 1, \dots, m$. The following calculation is an application of the product rule in

the Brownian motion case:

$$\begin{aligned}
D(XY) &= \sum_{i,j} D(F_i G_j v_i w_j) \\
&= \sum_{i,j} (DF_i G_j) \otimes (v_i w_j) \\
&= \sum_{i,j} F_i DG_j \otimes v_i w_j + \sum_{i,j} G_j DF_i \otimes v_i w_j \\
&= \sum_{i,j} F_i v_i (DG_j \otimes w_j) + \sum_{i,j} G_j w_j (DF_i \otimes v_i) \\
&= XDY + YDX.
\end{aligned} \tag{14}$$

□

The chain rule then can be generalized to differentiable functionals on $\mathbb{D}^{1,2}(L^2(\Omega_J))$ as shown in [5].

Theorem 3.3. *Let $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives and let $X = (X^{(1)}, \dots, X^{(m)})$ be a vector of random variables s.t. $X^{(j)} \in \mathbb{D}^{1,2}(L^2(\Omega_J))$ for all $j = 1, \dots, m$. Then we have*

$$D\Phi(X) = \sum_{i=1}^m \frac{\partial}{\partial x_i} \Phi(X) DX^{(i)}. \tag{15}$$

Definition 3.4. The **divergence operator**

$$\delta : L^2([0, T] \times \Omega_W, L^2(\Omega_J)) \rightarrow L^2(\Omega_W, L^2(\Omega_J)) \tag{16}$$

is the dual operator of $D : L^2(\Omega_W, L^2(\Omega_J)) \rightarrow L^2([0, T] \times \Omega_W, L^2(\Omega_J))$. Its domain $\text{dom}(\delta)$ is the set of $u \in L^2(\Omega_W, L^2(\Omega_J))$ s.t. there is a $c \in \mathbb{R}$ s.t.

$$E_{\Omega_W \times \Omega_J} \left[\int_0^T D_t X u_t dt \right] \leq c \|X\|_{L^2(\Omega_W \times \Omega_J)} \tag{17}$$

for all $X \in \mathbb{D}^{1,2}$. The divergence operator is characterized by

$$E_{\Omega_W \times \Omega_J} \left[\int_0^T D_t X u_t dt \right] = E[X \delta(u)] \tag{18}$$

for all $X \in \mathbb{D}^{1,2}$.

From the Riesz representation theorem follows, that (17) defines the greatest possible domain on which δ can be defined. The dual operator is also sometimes referred to as the **adjoint operator** of δ . Since the domain of D is dense in $L^2(\Omega_W \times \Omega_J)$, this operator is well-defined.

Proposition 3.5. *The operator δ is unbounded and closed, and $\text{dom}(\delta)$ is dense in $L^2(\Omega_W, L^2(\Omega_J))$.*

We now show, how a scalar real valued random variable can be factored out of the divergence. This will enable us to find explicit representations of the divergence in the second chapter.

Corollary 3.6. Let $X \in \mathbb{D}^{1,2}(L^2(\Omega_J))$ and $u \in \text{dom } \delta$ such that

$$Xu \in L^2([0, T] \times \Omega_W, L^2(\Omega_J)) \quad \text{and} \quad X\delta(u) - \int_0^T D_t X u_t dt \in L^2(\Omega_W, L^2(\Omega_J)). \quad (19)$$

Then $Xu \in \text{dom } \delta$ and

$$\delta(Xu) = X\delta(u) - \int_0^T (D_t X) u_t dt. \quad (20)$$

Proof. Let $Y \in \mathbb{D}^{1,2}(L^2(\Omega_J))$. We have to show, that

$$E \left[\int_0^T (D_t Y) X u_t dt \right] \leq \|X\delta(u) - \int_0^T (D_t X) u_t dt\|_{L^2(\Omega_W \times \Omega_J)} \|Y\|_{L^2(\Omega_W \times \Omega_J)} \quad (21)$$

and

$$E \left[\int_0^T (D_t Y) X u_t dt \right] = E \left[Y \left(X\delta(u) - \int_0^T (D_t X) u_t dt \right) \right] \quad (22)$$

hold. The product rule in Proposition 3.2 and the definition of the divergence operator give us

$$\begin{aligned} E \left[\int_0^T (D_t Y) X u_t dt \right] &= E \left[\int_0^T ((D_t X Y) - Y (D_t X)) u_t dt \right] \\ &= E \left[\int_0^T (D_t X Y) u_t dt \right] - E \left[\int_0^T Y (D_t X) u_t dt \right] \\ &= E[X Y \delta(u)] - E \left[\int_0^T Y (D_t X) u_t dt \right] \\ &= E \left[Y \left(X\delta(u) - \int_0^T (D_t X) u_t dt \right) \right] \\ &\leq \|Y\|_{L^2(\Omega_W \times \Omega_J)} \left\| X\delta(u) - \int_0^T (D_t X) u_t dt \right\|_{L^2(\Omega_W \times \Omega_J)}, \end{aligned} \quad (23)$$

equations (21) and (22) are indeed true. \square

4 The Calculation of Greeks in the Jump Diffusion Model

Let the underlying asset be described by an exponential Lévy process $X_t = x \exp(\tilde{X}_t)$ with non-vanishing Brownian motion part s.t. $X_0 = x$. X_t can therefore be represented as

$$S_t \exp \left(\tilde{\gamma} t + \sum_{i=1}^{N_t} \alpha Y_i + X_t^{(3)} \right). \quad (24)$$

Here, $S_t = x \exp(\mu t + \sigma W_t)$, $\mu = r - \sigma^2/2$ is the process from the Black-Scholes model, $\sum_{i=1}^{N_t} \alpha Y_i$ is a compound Poisson process where λ denotes the intensity of the Poisson process N_t and $X_t^{(3)}$ is a square integrable pure jump martingale that almost surely has a countable number of jumps on finite intervals.

Since $\exp(\gamma t + \sigma W_t) \in \mathbb{D}^{1,2}$, X_t is Malliavin derivable and we have

$$DX_t = D \exp(\gamma t + \sigma W_t) \otimes \exp(X_t^J) = \sigma \exp(\gamma t + \sigma W_t) \mathbf{1}_{[0,t]} \otimes \exp(X_t^J) = \sigma X_t \mathbf{1}_{[0,t]}. \quad (25)$$

4.1 The Calculation of European Greeks in the Jump Diffusion Model

As in the Brownian motion case, the **integration by parts formula** from [4] is the main tool for the calculation of European Greeks. It can easily be generalized to the Lévy process case. We state this formula and give a proof in this generalized setting. 5t

Theorem 4.1 (The integration by parts formula). *Let G be a real valued random variable, $F \in \mathbb{D}^{1,2}(L^2(\Omega_J))$ and let $u \in L^2([0, T] \times \Omega)$ such that*

- $\int_0^T D_t F u_t dt \neq 0$ a.s. and
- $Gu(\int_0^T D_t F u_t dt)^{-1} \in \text{dom } \delta$.

If $\Phi \in C^1(\mathbb{R})$ has a bounded derivative we have

$$E[\Phi'(F)G] = E \left[\Phi(F) \delta \left(\frac{u_t G}{\int_0^T D_t F u_t dt} \right) \right]. \quad (26)$$

Proof. With the chain rule in Propostition 3.3 and the characterization of the Skorohod integral (18) we get

$$\begin{aligned} E[\Phi'(F)G] &= E \left[\int_0^T \Phi'(F) D_t F u_t dt \frac{G}{\int_0^T D_t F u_t dt} \right] = E \left[\int_0^T D_t \Phi(F) \frac{u_t G}{\int_0^T D_s F u_s ds} dt \right] \\ &= E \left[\Phi(F) \delta \left(\frac{u_t G}{\int_0^T D_t F u_t dt} \right) \right]. \end{aligned} \quad (27)$$

□

We now present closed formulas for European Greeks.

Theorem 4.2. *Let $\Phi \in L^2(\mathbb{R}_{[0,\infty)})$ be bounded on compacts with a finite number of jumps and without jump discontinuities. Then the Greeks for European options can be represented as*

$$\Delta = \frac{\partial V_0}{\partial x} = \frac{e^{-rT}}{x\sigma T} E[\Phi(X_T)W_T] \quad (28)$$

$$\mathcal{V} = \frac{\partial V_0}{\partial \sigma} = e^{-rT} E \left[\Phi(X_T) \left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma} \right) \right] \quad (29)$$

$$\rho = \frac{\partial V_0}{\partial r} = T e^{-rT} E \left[\Phi(X_T) \left(\frac{W_T}{\sigma T} - 1 \right) \right] \quad (30)$$

$$\Theta = -\frac{\partial V_0}{\partial T} = e^{-rT} E \left[\Phi(X_T) \left(\frac{W_T^2}{2T^2} + \mu \frac{W_T}{\sigma T} - \left(\frac{1}{2T} + r \right) \right) \right] \quad (31)$$

$$\Gamma = \frac{\partial V_0}{\partial x^2} = \frac{e^{-rT}}{x^2 \sigma T} E \left[\Phi(X_T) \left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma} \right) \right] \quad (32)$$

$$A = \frac{\partial V_0}{\partial \alpha} = \frac{e^{-rT}}{\sigma T} E \left[\Phi(X_T) W_T \sum_{i=1}^{N_T} Y_i \right] \quad (33)$$

The proofs of (28) - (32) are similar to the proofs in the Black Scholes case, see for example [3]. We will exemplarily give the proof for (28). The Greek A (capital Alpha) is a Greek that does not exist in the Black Scholes model, we also give its derivation here.

Proof of (28). We consider X_T as a function $X_T : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$, $x \mapsto X_T(x) = x \exp(\tilde{X}_T)$ of the initial value. Let $\Phi \in C^1(\mathbb{R})$ be a continuously differentiable function with bounded derivative. Since $\frac{\partial}{\partial x} X_T = \frac{1}{x} X_T$ we have

$$\frac{\partial}{\partial x} E[e^{-rT} \Phi(X_T)] = \frac{e^{-rT}}{x} E[\Phi'(X_T) X_T]. \quad (34)$$

We apply the integration by parts formula in Theorem 4.1 with $F = G = X_T$ and $u = DX_T$ which gives us

$$E[\Phi'(X_T) X_T] = E \left[\Phi(X_T) \delta \left(\frac{(DX_T) X_T}{\int_0^T (D_t X_T)^2} \right) \right] = \frac{1}{\sigma T} E[\Phi(X_T) \delta(\mathbf{1}_{[0,T]})] = \frac{1}{\sigma T} E[\Phi(X_T) W_T]. \quad (35)$$

We now generalize (28) to the case, where $\Phi \in L^2(\mathbb{R}_{\geq 0})$ is bounded on compacts, with a finite number of jumps and without jump discontinuities. To this end, we have to find a sequence (Φ_n) of continuously differentiable functions with bounded derivatives s.t. $E[e^{-rT} \Phi_n(X_T(x))]$ converges to $E[e^{-rT} \Phi(X_T(x))]$ for some $x > 0$ and s.t. the function

$$x \mapsto \frac{e^{-rT}}{x \sigma T} E[\Phi_n(X_T(x)) W_T] \quad (36)$$

converges uniformly to

$$x \mapsto \frac{e^{-rT}}{x \sigma T} E[\Phi(X_T(x)) W_T]. \quad (37)$$

Let $0 \leq x_1 \leq \dots \leq x_l < \infty$ be the points of discontinuity of Φ . Define the sets

$$A_n = \cup_{i=1}^l (\min 0, x_i - 1/n, x_i) \quad (38)$$

Let $(\Phi_n) \in C_b^1(\mathbb{R}_{\geq 0})$ be a sequence of continuously differentiable functions with bounded derivatives s.t.

$$\Phi_n|_{A_n^c} \equiv \Phi|_{A_n^c} \text{ and } \Phi_n \leq \Phi + \frac{1}{n} \quad (39)$$

for all $n \in \mathbb{N}$. Then $E[\Phi_n(X_T)]$ converges to $E[\Phi(X_T)]$ for all $x > 0$ as $n \rightarrow \infty$. With the Cauchy-Schwarz inequality we get

$$E[(\Phi_n - \Phi)(X_T) W_T]^2 \leq E[(\Phi_n - \Phi)^2(X_T)] E[W_T^2]. \quad (40)$$

From the continuity of $x \mapsto \frac{e^{-rT}}{x \sigma T} E[(\Phi_n - \Phi)^2(X_T)(x)]$ it follows that for every compact set $K \subseteq (0, \infty)$ there is a $\tilde{x} \in K$ s.t.

$$\sup_{x \in K} \frac{e^{-rT}}{x \sigma T} E[(\Phi_n - \Phi)(X_T(x)) W_T]^2 \leq \frac{e^{-rT}}{\tilde{x} \sigma T} E[(\Phi_n - \Phi)^2(X_T(\tilde{x}))] E[W_T^2] \xrightarrow{n \rightarrow \infty} 0. \quad (41)$$

We conclude that the function (36) converges uniformly to the function (37) and that (28) holds

for $\Phi \in L^2(\mathbb{R}_{\geq 0})$ which is bounded on compacts, with a finite number of jumps and without jump discontinuities.

Proof of (33). Let $\Phi \in C^1(\mathbb{R}_{[0,\infty)})$. The derivative of X_T with respect to α is $\frac{\partial}{\partial \alpha} X_T = X_T \sum_{i=1}^{N_T} Y_i$. We apply the integration by parts formula in Theorem 4.1 with $u = \mathbf{1}_{[0,T]}$:

$$\begin{aligned} \frac{\partial}{\partial \alpha} E[\Phi(X_T)] &= E\left[\Phi'(X_T) X_T \sum_{i=1}^{N_T} Y_i\right] \\ &= E\left[\Phi(X_T) \delta\left(\frac{X_T \sum_{i=1}^{N_T} Y_i}{\sigma T X_T}\right)\right] \\ &= \frac{1}{\sigma T} E\left[\Phi(X_T) \delta\left(\sum_{i=1}^{N_T} Y_i\right)\right] \\ &= \frac{1}{\sigma T} E\left[\Phi(X_T) W_T \sum_{i=1}^{N_T} Y_i\right]. \end{aligned} \tag{42}$$

The generalization to more general $\Phi \in L^2(\mathbb{R}_{[0,\infty)})$ follows then analogously to the proof of (28). \square

4.2 The Calculation of Asian Greeks in the Jump Diffusion model

Asian options are options, where the payoff is determined by the average price of the underlying asset. An Asian call option with exercise time T and exercise price K has the payoff function

$$\Phi_c = \Phi_c\left(\frac{1}{T} \int_0^T X_t dt\right) = \left(\frac{1}{T} \int_0^T X_t dt - K\right)^+, \tag{43}$$

an Asian put option has the payoff function

$$\Phi_p = \Phi_p\left(\frac{1}{T} \int_0^T X_t dt\right) = \left(K - \frac{1}{T} \int_0^T X_t dt\right)^+. \tag{44}$$

The option price for an Asian option then is given by

$$V_0 = E\left[e^{-rT} \Phi\left(\frac{1}{T} \int_0^T X_t dt\right)\right]. \tag{45}$$

We introduce

$$I_{(n)} = \int_0^T t^n X_t dt, \quad n \geq 0. \tag{46}$$

Before proceeding we present two useful lemmas:

Lemma 4.3. The random variables $I_{(n)} = \int_0^T t^n X_t dt$ are in $\mathbb{D}^{1,2}(L^2(\Omega_J))$ and we have

$$D_s I_{(n)} = D_s \int_0^T t^n X_t dt = \sigma \int_s^T t^n X_t dt, \tag{47}$$

for all $s \in [0, T]$.

Proof. This follows with a approximation of a monotonously increasing sequence of Riemann sums. We define the Riemann sums

$$S_n^k(X) = \frac{T}{2^k} \sum_{i=0}^{2^k-1} s_{i,k}^{(n)} X_{s_{i,k}}, \quad (48)$$

where we set $s_{0,0} = 0$ and

$$s_{i,k+1} = \begin{cases} s_{\lfloor i/2 \rfloor, k} & \text{if } s_{\lfloor i/2 \rfloor, k} \leq \frac{Ti}{2^{k+1}} \\ \frac{Ti}{2^{k+1}} & \text{else} \end{cases} \quad (49)$$

for $i \leq 2^{k+1}$. The sequence $S_{(n)}^k(X)$ is clearly in $\mathbb{D}^{1,2}(L^2(\Omega_J))$ and converges to $I_{(n)}$ a.s. Since the sequence $(S_{(n)}^k(X))_k$ is also monotonously increasing it follows, that $S_{(n)}^k(X)$ converges to $I_{(n)}$ in $L^2(\Omega)$. The derivative of $S_{(n)}^k(X)$ is

$$DS_{(n)}^k(X) = \frac{T}{2^k} \sum_{i=0}^{2^k-1} Ds_i^k X_{s_i} = \frac{\sigma T}{2^k} \sum_{i=0}^{2^k-1} s_i^k X_{s_i} \mathbf{1}_{[0, s_i]}, \quad (50)$$

$D_s S_{(n)}^k(X)$ is monotonously increasing and therefore converges also to $\int_s^T t^k X_t dt$ in $L^2(\Omega)$. Therefore we conclude that $\int_0^T X_t dt \in \mathbb{D}^{1,2}(L^2(\Omega))$ and that

$$D_s \int_0^T t^n X_t dt = \sigma \int_s^T t^n X_t dt \quad (51)$$

holds. \square

A straightforward application of the integration by parts formula gives us the following iterative formula:

Lemma 4.4. We have

$$\int_0^T D_s I_{(n)} ds = \sigma I_{(n+1)} \quad (52)$$

for all $n \geq 0$.

Now we can finally obtain the integration by parts formula which will be the key for the calculation of Asian Greeks:

Corollary 4.5. Let Φ be a continuously differentiable payoff function with bounded derivatives and let F be a random variable s.t. $\frac{F}{I_{(1)}}$ is Skorohod integrable. Then we have

$$E\left[\Phi'\left(\frac{I_{(0)}}{T}\right)F\right] = E\left[\Phi\left(\frac{I_{(0)}}{T}\right)\delta\left(\frac{TF}{\sigma I_{(1)}}\right)\right]. \quad (53)$$

Proof. With Lemmas 4.3 and 4.4 we get

$$\int_0^T D_s \Phi \left(\frac{I_{(0)}}{T} \right) ds = \int_0^T \Phi' \left(\frac{I_{(0)}}{T} \right) \frac{\sigma}{T} \left(\int_s^T X_t dt \right) ds = \frac{\sigma}{T} \Phi' \left(\frac{I_{(0)}}{T} \right) I_{(1)}. \quad (54)$$

The result then follows from the definition of the Skorohod integral. \square

Theorem 4.6. *Let $\Phi \in L^2(\mathbb{R}_{[0,\infty)})$ be bounded on compacts with a finite number of jumps and without jump discontinuities. Then the Greeks for Asian options are given by*

$$\mathcal{G} = e^{-rT} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \pi_{\mathcal{G}} \right], \quad \mathcal{G} \in \{\Delta, \mathcal{V}, \rho, \Theta, \Gamma, A\} \quad (55)$$

where

$$\pi_{\Delta} = \frac{1}{\sigma x} \left(-\sigma + \frac{I_{(0)}}{I_{(1)}} W_T + \sigma \frac{I_{(0)} I_{(2)}}{I_{(1)}^2} \right) \quad (56)$$

$$\pi_{\mathcal{V}} = \frac{1}{\sigma} \left(-(1 + \sigma W_T) + \frac{W_T \int_0^T X_t W_t dt - \sigma \int_0^T t X_t W_t dt}{I_{(1)}} + \frac{\sigma (\int_0^T X_t W_t dt) I_{(2)}}{I_{(1)}^2} \right) \quad (57)$$

$$\pi_{\rho} = \left(\frac{W_T}{\sigma} - T \right) \quad (58)$$

$$\pi_{\Theta} = r - \frac{1}{T} + \frac{\frac{1}{\sigma T} I_{(0)} W_T - \frac{1}{\sigma} X_T W_T - T X_T}{I_{(1)}} + \frac{\frac{1}{T} I_{(0)} I_{(2)} + I_{(2)} W_T}{I_{(1)}^2} \quad (59)$$

$$\pi_{\Gamma} = \frac{1}{\sigma x^2} \left((\sigma + \sigma^2) - \frac{(\sigma - \sigma W_T - W_T) I_{(0)}}{I_{(1)}} + \frac{-\sigma I_{(0)} I_{(2)} + W_T^2 I_{(0)}^2 - 3\sigma^2 I_{(0)} I_{(2)}}{I_{(1)}^2} \right) \quad (60)$$

$$+ \frac{\sigma W_T I_{(0)}^2 I_{(2)} - \sigma^2 I_{(0)}^2 I_{(3)} + 2\sigma I_{(0)}^2 I_{(2)}}{I_{(1)}^3} + \frac{3\sigma^2 I_{(0)}^2 I_{(2)}}{I_{(1)}^4} \quad (61)$$

$$\pi_A = \frac{1}{\alpha} \left(\frac{\frac{1}{\sigma} W_T \int_0^T X_t^{(2)} X_t dt - \int_0^T t X_t^{(2)} X_t dt}{I_{(1)}} + \frac{\int_0^T X_t^{(2)} X_t dt I_{(2)}}{I_{(1)}^2} \right). \quad (62)$$

We give the proofs of (56) and (62); the proofs (57) - (61) can be found in the appendix. Throughout the proofs, Φ will be a continuously differentiable function with bounded derivative. The generalization to more general $\Phi \in L^2(\mathbb{R}_{[0,\infty)})$ follows then like in the proof of (28).

Proof of (56). With Corollary 4.5 we have

$$\Delta = \frac{\partial}{\partial x} E \left[e^{-rT} \Phi \left(\frac{I_{(0)}}{T} \right) \right] = e^{-rT} E \left[\Phi' \left(\frac{I_{(0)}}{T} \right) \frac{I_{(0)}}{xT} \right] = \frac{e^{-rT}}{\sigma x} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \delta \left(\frac{I_{(0)}}{I_{(1)}} \right) \right]. \quad (63)$$

By Corollary 3.6

$$\delta \left(\frac{I_{(0)}}{I_{(1)}} \right) = \frac{I_{(0)}}{I_{(1)}} \delta(\mathbf{1}_{[0,T]}) - \int_0^T D_s \frac{I_{(0)}}{I_{(1)}} ds. \quad (64)$$

We apply the chain rule to the second term of the right side of (64), and we obtain from Lemma

4.4:

$$\begin{aligned}
\int_0^T D_s \frac{I_{(0)}}{I_{(1)}} ds &= \int_0^T \frac{I_{(1)} D_s I_{(0)} - I_{(0)} D_s I_{(1)}}{I_{(1)}^2} ds \\
&= \frac{\int_0^T D_s I_{(0)} ds}{I_{(1)}} - \frac{I_{(0)} \int_0^T D_s I_{(1)} ds}{I_{(1)}^2} \\
&= \sigma - \frac{\sigma I_{(0)} I_{(2)}}{I_{(1)}^2}.
\end{aligned} \tag{65}$$

We therefore can express the divergence as

$$\delta \left(\frac{I_{(0)}}{I_{(1)}} \right) = \frac{I_{(0)}}{I_{(1)}} W_T - \sigma + \frac{\sigma I_{(0)} I_{(2)}}{I_{(1)}^2}. \tag{66}$$

This finally gives us a closed expression for Δ :

$$\Delta = \frac{e^{-rT}}{\sigma x} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \left(-\sigma + \frac{I_{(0)}}{I_{(1)}} W_T + \sigma \frac{I_{(0)} I_{(2)}}{I_{(1)}^2} \right) \right]. \tag{67}$$

Proof of (62). The derivative of X_t with respect to α is

$$\frac{\partial}{\partial \alpha} X_t = X_t \frac{\partial}{\partial \alpha} \left(X_t^{(1)} + \sum_{i=1}^{N_t} \alpha Y_i + X_t^{(3)} \right) = X_t \sum_{i=1}^{N_t} Y_i = \frac{1}{\alpha} X_t^{(2)} X_t. \tag{68}$$

This gives us

$$\begin{aligned}
\frac{\partial}{\partial \alpha} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \right] &= E \left[\Phi' \left(\frac{I_{(0)}}{T} \right) \frac{1}{\alpha T} \int_0^T X_t^{(2)} X_t dt \right] = \frac{1}{\alpha} E \left[\frac{\sigma}{T} \Phi' \left(\frac{I_{(0)}}{T} \right) I_{(1)} \left(\frac{\int_0^T X_t^{(2)} X_t dt}{\sigma I_{(1)}} \right) \right] \\
&= \frac{1}{\alpha} E \left[\int_0^T D_s \Phi \left(\frac{I_{(0)}}{T} \right) \left(\frac{\int_0^T X_t^{(2)} X_t dt}{\sigma I_{(1)}} \right) ds \right] \\
&= \frac{1}{\alpha} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \delta \left(\frac{\int_0^T X_t^{(2)} X_t dt}{\sigma I_{(1)}} \right) \right].
\end{aligned} \tag{69}$$

We calculate the Skorohod integral with Corollary 3.6:

$$\delta \left(\frac{\int_0^T X_t^{(2)} X_t dt}{\sigma I_{(1)}} \right) = \frac{\int_0^T X_t^{(2)} X_t dt}{\sigma I_{(1)}} W_T - \frac{1}{\sigma} \int_0^T D_s \frac{\int_0^T X_t^{(2)} X_t dt}{I_{(1)}} ds. \tag{70}$$

The second term can be rewritten as

$$\begin{aligned}
\frac{1}{\sigma} \int_0^T D_s \frac{\int_0^T X_t^{(2)} X_t dt}{I_{(1)}} ds &= \frac{1}{\sigma} \left(\frac{\int_0^T \int_s^T \sigma X_t^{(2)} X_t dt ds}{I_{(1)}} - \frac{\int_0^T X_t^{(2)} X_t dt \int_0^T D_s I_{(1)} ds}{I_{(1)}^2} \right) \\
&= \frac{\int_0^T t X_t^{(2)} X_t dt}{I_{(1)}} - \frac{\int_0^T X_t^{(2)} X_t dt I_{(2)}}{I_{(1)}^2}.
\end{aligned} \tag{71}$$

As a consequence we obtain,

$$A = \frac{1}{\alpha} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \left(\frac{\frac{1}{\sigma} W_T \int_0^T X_t^{(2)} X_t dt - \int_0^T t X_t^{(2)} X_t dt}{I_{(1)}} + \frac{\int_0^T X_t^{(2)} X_t dt I_{(2)}}{I_{(1)}^2} \right) \right]. \quad (72)$$

□

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5 Appendix

In the appendix we give the proofs of (57) ot (61). Throughout, Φ will be a continuously differentiable function with bounded derivative. The generalization to the payoff functions of put- and call options follows then like in the proof of (28).

Proof of (57). The derivative of X_t with respect to σ is

$$\frac{\partial}{\partial \sigma} X_t = X_t(W_t - \sigma t). \quad (73)$$

This gives us

$$\begin{aligned} \mathcal{V} &= \frac{\partial}{\partial \sigma} \frac{I_{(0)}}{T} = \frac{1}{T} \int_0^T X_t(W_t - \sigma t) dt \\ &= \frac{1}{T} \int_0^T X_t W_t dt - \frac{\sigma}{T} I_{(1)}. \end{aligned} \quad (74)$$

With this presentation we can write \mathcal{V} as

$$\begin{aligned} \mathcal{V} &= \frac{\partial}{\partial \sigma} E \left[e^{-rT} \Phi \left(\frac{I_{(0)}}{T} \right) \right] \\ &= e^{-rT} E \left[\underbrace{\frac{1}{T} \Phi' \left(\frac{I_{(0)}}{T} \right) \left(\int_0^T X_t W_t dt \right)}_{(*)} \right] - e^{-rT} E \left[\underbrace{\frac{\sigma}{T} \Phi' \left(\frac{I_{(0)}}{T} \right) I_{(1)}}_{(**)} \right]. \end{aligned} \quad (75)$$

Corollary 4.5 then allows us to calculate

$$(*) = \frac{1}{\sigma} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \delta \left(\frac{\int_0^T X_t W_t dt}{I_{(1)}} \right) \right] \quad (76)$$

and

$$(**) = E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \delta(\mathbf{1}_{(0,T]}) \right] = E \left[\Phi \left(\frac{I_{(0)}}{T} \right) W_T \right]. \quad (77)$$

With Corollary 3.6 we get

$$\begin{aligned} \delta \left(\mathbf{1}_{[0,T]} \frac{\int_0^T X_t W_t dt}{I_{(1)}} \right) &= \left(\frac{\int_0^T X_t W_t dt}{I_{(1)}} \right) \int_0^T \mathbf{1}_{[0,T]}(t) dW_t - \int_0^T D_s \left(\frac{\int_0^T X_t W_t dt}{I_{(1)}} \right) ds \\ &= \left(\frac{\int_0^T X_t W_t dt}{I_{(1)}} \right) W_T - \int_0^T D_s \left(\frac{\int_0^T X_t W_t dt}{I_{(1)}} \right) ds. \end{aligned} \quad (78)$$

With

$$D_s X_t W_t = X_t(\sigma W_t + 1) \quad (79)$$

for $s \leq t$ it follows that

$$D_s \int_0^T X_t W_t dt = \sigma \int_s^T X_t W_t dt + \int_s^T X_t dt. \quad (80)$$

From the chain rule we obtain

$$\begin{aligned} & \int_0^T D_s \left(\frac{\int_0^T X_t W_t dt}{I_{(1)}} \right) ds \\ &= \frac{I_{(1)} (\sigma \int_0^T (\int_s^T X_t W_t dt) ds + \int_0^T (\int_s^T X_t dt) ds) - (\int_0^T X_t W_t dt) \int_0^T D_s I_{(1)} ds}{I_{(1)}^2} \\ &= \frac{\sigma \int_0^T t X_t W_t dt}{I_{(1)}} + 1 - \sigma \frac{(\int_0^T X_t W_t dt) I_{(2)}}{I_{(1)}^2}. \end{aligned} \quad (81)$$

We therefore can write the divergence as

$$\begin{aligned} & \delta \left(\mathbf{1}_{[0,T]} \frac{\int_0^T X_t W_t dt}{I_{(1)}} \right) \\ &= \left(\frac{\int_0^T X_t W_t dt}{I_{(1)}} \right) W_T - \frac{\sigma \int_0^T t X_t W_t dt}{I_{(1)}} - 1 + \sigma \frac{(\int_0^T X_t W_t dt) I_{(2)}}{I_{(1)}^2}. \end{aligned} \quad (82)$$

This implies

$$(*) = \frac{1}{\sigma} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \left(-1 + \frac{W_T \int_0^T X_t W_t dt - \sigma \int_0^T t X_t W_t dt}{I_{(1)}} + \frac{\sigma (\int_0^T X_t W_t dt) I_{(2)}}{I_{(1)}^2} \right) \right]. \quad (83)$$

We conclude that

$$\begin{aligned} \mathcal{V} = \frac{e^{-rT}}{\sigma} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \left(- (1 + \sigma W_T) + \frac{W_T \int_0^T X_t W_t dt - \sigma \int_0^T t X_t W_t dt}{I_{(1)}} \right. \right. \\ \left. \left. + \frac{\sigma (\int_0^T X_t W_t dt) I_{(2)}}{I_{(1)}^2} \right) \right]. \end{aligned} \quad (84)$$

Proof of (58). We have $\frac{\partial}{\partial r} X_t = t X_t$ and therefore $\frac{\partial}{\partial r} I_{(0)} = I_{(1)}$. As consequence we get

$$\rho = \frac{\partial V_0}{\partial r} = \frac{\partial}{\partial r} E \left[e^{-rT} \Phi \left(\frac{I_{(0)}}{T} \right) \right] = -T e^{-rT} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \right] + e^{-rT} E \left[\Phi' \left(\frac{I_{(0)}}{T} \right) \frac{I_{(1)}}{T} \right]. \quad (85)$$

From Corollary 4.5 we get

$$E \left[\Phi' \left(\frac{I_{(0)}}{T} \right) \frac{I_{(1)}}{T} \right] = \frac{1}{\sigma} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) W_T \right], \quad (86)$$

and conclude that

$$\mathcal{V} = e^{-rT} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \left(\frac{W_T}{\sigma} - T \right) \right]. \quad (87)$$

Proof of (59). A straightforward calculation gives us

$$\begin{aligned}\Theta &= -\frac{\partial V_0}{\partial T} = -\frac{\partial}{\partial T} E\left[e^{-rT} \Phi\left(\frac{I_{(0)}}{T}\right)\right] \\ &= r e^{-rT} E\left[\Phi\left(\frac{I_{(0)}}{T}\right)\right] + e^{-rT} E\left[\Phi'\left(\frac{I_{(0)}}{T}\right) \left(\frac{I_{(0)}}{T^2} - \frac{1}{T} X_T\right)\right].\end{aligned}\quad (88)$$

Corollary 4.5 gives us

$$E\left[\Phi'\left(\frac{I_{(0)}}{T}\right) \left(\frac{I_{(0)}}{T^2} - \frac{1}{T} X_T\right)\right] = E\left[\Phi\left(\frac{I_{(0)}}{T}\right) \delta\left(\frac{\frac{1}{T} I_{(0)} - X_T}{\sigma I_{(1)}}\right)\right]. \quad (89)$$

To calculate the Skorohod integral in the expectation of the right side of (89) apply Corollary 3.6:

$$\delta\left(\frac{\frac{1}{T} I_{(0)} - X_T}{\sigma I_{(1)}}\right) = \frac{\frac{1}{T} I_{(0)} - X_T}{\sigma I_{(1)}} W_T - \int_0^T D_s \left(\frac{\frac{1}{T} I_{(0)} - X_T}{\sigma I_{(1)}}\right) ds. \quad (90)$$

With the chain rule, Lemma 4.3 and because $\int_0^T \int_s^T X_t dt ds = \int_0^T t X_t dt$ we get

$$\begin{aligned}\int_0^T D_s \left(\frac{\frac{1}{T} I_{(0)}}{\sigma I_{(1)}}\right) ds &= \int_0^T \frac{\frac{\sigma^2}{T} I_{(1)} \int_s^T X_t dt - \frac{\sigma^2}{T} I_{(0)} \int_s^T t X_t dt}{\sigma^2 I_{(1)}^2} ds \\ &= \frac{1}{T} - \frac{I_{(0)} I_{(2)}}{T I_{(1)}^2},\end{aligned}\quad (91)$$

and in the same manner

$$\begin{aligned}\int_0^T D_s \frac{X_T}{\sigma I_{(1)}} ds &= \int_0^T \frac{\sigma^2 I_{(1)} X_T - \sigma^2 \int_s^T t X_t dt X_T}{\sigma^2 I_{(1)}^2} ds \\ &= \frac{T X_T}{I_{(1)}} - \frac{I_{(2)} X_T}{I_{(1)}^2}.\end{aligned}\quad (92)$$

Together, this leads to

$$\begin{aligned}\delta\left(\frac{\frac{1}{T} I_{(0)} - X_T}{\sigma I_{(1)}}\right) &= \frac{\frac{1}{T} I_{(0)} - X_T}{\sigma I_{(1)}} W_T - \frac{1}{T} + \frac{I_{(0)} I_{(2)}}{T I_{(1)}^2} - \frac{T X_T}{I_{(1)}} + \frac{I_{(2)} X_T}{I_{(1)}^2} \\ &= -\frac{1}{T} + \frac{\frac{1}{\sigma T} I_{(0)} W_T - \frac{1}{\sigma} X_T W_T - T X_T}{I_{(1)}} + \frac{\frac{1}{T} I_{(0)} I_{(2)} + I_{(2)} W_T}{I_{(1)}^2}.\end{aligned}\quad (93)$$

We can finally write

$$\Theta = e^{-rT} E\left[\Phi\left(\frac{I_{(0)}}{T}\right) \left(r - \frac{1}{T} + \frac{\frac{1}{\sigma T} I_{(0)} W_T - \frac{1}{\sigma} X_T W_T - T X_T}{I_{(1)}} + \frac{\frac{1}{T} I_{(0)} I_{(2)} + I_{(2)} W_T}{I_{(1)}^2}\right)\right]. \quad (94)$$

□

Proof of (61). To calculate Γ we have to differentiate

$$\Delta = \frac{e^{-rT}}{\sigma x} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \left(-\sigma + \frac{I_{(0)}}{I_{(1)}} W_T + \sigma \frac{I_{(0)} I_{(2)}}{I_{(1)}^2} \right) \right] \quad (95)$$

with respect to the initial value x again: This gives

$$\begin{aligned} \Gamma &= \frac{\partial}{\partial x} \Delta \\ &= -\frac{1}{x} \Delta + \frac{e^{-rT}}{\sigma x} E \left[\Phi' \left(\frac{I_{(0)}}{T} \right) \frac{I_{(0)}}{xT} \left(-\sigma + \frac{I_{(0)}}{I_{(1)}} W_T + \sigma \frac{I_{(0)} I_{(2)}}{I_{(1)}^2} \right) \right] \\ &\quad + \frac{e^{-rT}}{\sigma x} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \frac{\partial}{\partial x} \left(-\sigma + \frac{I_{(0)}}{I_{(1)}} W_T + \sigma \frac{I_{(0)} I_{(2)}}{I_{(1)}^2} \right) \right] \\ &= -\frac{1}{x} \Delta + \frac{e^{-rT}}{\sigma x} E \left[\Phi' \left(\frac{I_{(0)}}{T} \right) \frac{I_{(0)}}{xT} \left(-\sigma + \frac{I_{(0)}}{I_{(1)}} W_T + \sigma \frac{I_{(0)} I_{(2)}}{I_{(1)}^2} \right) \right] \end{aligned} \quad (96)$$

since $\frac{I_{(0)}}{I_{(1)}}$ and $\frac{I_{(0)} I_{(2)}}{I_{(1)}^2}$ are constant functions of x . We apply Corollary 4.5 to the right side of (96) to get

$$\Gamma = -\frac{1}{x} \Delta + \frac{e^{-rT}}{\sigma^2 x^2} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \delta \left(-\sigma \frac{I_{(0)}}{I_{(1)}} + \frac{I_{(0)}^2}{I_{(1)}^2} W_T + \sigma \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^3} \right) \right]. \quad (97)$$

We apply Corollary 3.6 to the Skorohod integral of the right hand side of (97) with $F = \mathbf{1}_{[0,T]}$:

$$\begin{aligned} &\delta \left(-\sigma \frac{I_{(0)}}{I_{(1)}} + \frac{I_{(0)}^2}{I_{(1)}^2} W_T + \sigma \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^3} \right) \\ &= \left(-\sigma \frac{I_{(0)}}{I_{(1)}} + \frac{I_{(0)}^2}{I_{(1)}^2} W_T + \sigma \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^3} \right) \delta(\mathbf{1}_{[0,T]}) - \int_0^T D_s \left(-\sigma \frac{I_{(0)}}{I_{(1)}} + \frac{I_{(0)}^2}{I_{(1)}^2} W_T + \sigma \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^3} \right) ds \\ &= \frac{-\sigma W_T I_{(0)}}{I_{(1)}} + \frac{W_T^2 I_{(0)}^2}{I_{(1)}^2} + \frac{\sigma W_T I_{(0)}^2 I_{(2)}}{I_{(1)}^3} - \int_0^T D_s \left(-\sigma \frac{I_{(0)}}{I_{(1)}} + \frac{I_{(0)}^2}{I_{(1)}^2} W_T + \sigma \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^3} \right) ds. \end{aligned} \quad (98)$$

For the right hand expression of the right side of (98) we obtain

$$\begin{aligned} &\int_0^T D_s \left(-\sigma \frac{I_{(0)}}{I_{(1)}} + \frac{I_{(0)}^2}{I_{(1)}^2} W_T + \sigma \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^3} \right) ds \\ &= -\sigma \frac{I_{(1)} \sigma I_{(1)} - I_{(0)} \sigma I_{(2)}}{I_{(1)}^2} + \frac{I_{(1)}^2 2 I_{(0)} \sigma I_{(1)} - I_{(0)}^2 2 I_{(1)} \sigma I_{(2)}}{I_{(1)}^4} \\ &\quad + \sigma \frac{I_{(1)}^3 2 I_{(0)} \sigma I_{(1)} I_{(2)} + I_{(1)}^3 I_{(0)}^2 \sigma I_{(3)} - I_{(0)}^2 I_{(2)} 3 I_{(1)}^2 \sigma I_{(2)}}{I_{(1)}^6} \\ &= -\sigma^2 + \sigma^2 \frac{I_{(0)} I_{(2)}}{I_{(1)}^2} + 2\sigma \frac{I_{(0)}}{I_{(1)}} - 2\sigma \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^3} + 2\sigma^2 \frac{I_{(0)} I_{(2)}}{I_{(1)}^2} + \sigma^2 \frac{I_{(0)}^2 I_{(3)}}{I_{(1)}^3} - 3\sigma^2 \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^4} \\ &= -\sigma^2 + \frac{2\sigma I_{(0)}}{I_{(1)}} + \frac{3\sigma^2 I_{(0)} I_{(2)}}{I_{(1)}^2} + \frac{\sigma^2 I_{(0)}^2 I_{(3)}}{I_{(1)}^3} - 2\sigma \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^3} - \frac{3\sigma^2 I_{(0)}^2 I_{(2)}}{I_{(1)}^4} \end{aligned} \quad (99)$$

Therefore we can write left side of (100) as

$$\sigma^2 - \frac{\sigma(2 - W_T)I_{(0)}}{I_{(1)}} + \frac{W_T^2 I_{(0)}^2 - 3\sigma^2 I_{(0)} I_{(2)}}{I_{(1)}^2} + \frac{\sigma W_T I_{(0)}^2 I_{(2)} - \sigma^2 I_{(0)}^2 I_{(3)} + 2\sigma I_{(0)}^2 I_{(2)}}{I_{(1)}^3} + \frac{3\sigma^2 I_{(0)}^2 I_{(2)}^2}{I_{(1)}^4}. \quad (100)$$

As a consequence of (100) we finally obtain

$$\Gamma = \frac{e^{-rT}}{\sigma x^2} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \left(\sigma + \sigma^2 - \frac{(2\sigma - \sigma W_T - W_T)I_{(0)}}{I_{(1)}} + \frac{-\sigma I_{(0)} I_{(2)} + W_T^2 I_{(0)}^2 - 3\sigma^2 I_{(0)} I_{(2)}}{I_{(1)}^2} \right. \right. \\ \left. \left. + \frac{\sigma W_T I_{(0)}^2 I_{(2)} - \sigma^2 I_{(0)}^2 I_{(3)} + 2\sigma I_{(0)}^2 I_{(2)}}{I_{(1)}^3} + \frac{3\sigma^2 I_{(0)}^2 I_{(2)}^2}{I_{(1)}^4} \right) \right] \quad (101)$$

$$\int_0^T D_s \left(-\sigma \frac{I_{(0)}}{I_{(1)}} + \frac{I_{(0)}^2}{I_{(1)}^2} W_T + \sigma \frac{I_{(0)}^2 I_{(2)}}{I_{(1)}^3} \right) ds \\ = \int_0^T \frac{2I_{(0)}(D_s I_{(0)})I_{(2)} + I_{(0)}^2 D_s I_{(2)}}{I_{(1)}^3} - \frac{I_{(0)}^2 I_{(2)} 3I_{(1)}^2 (D_s I_{(1)})}{I_{(1)}^6} \\ + \frac{I_{(0)}^2 + W_T 2I_{(0)}(D_s I_{(0)})}{I_{(1)}^2} - \frac{W_T I_{(0)}^2 2I_{(1)} D_s I_{(1)}}{I_{(1)}^4} ds \\ = \frac{2I_{(0)}\sigma I_{(1)} I_{(2)} + I_{(0)}^2 \sigma I_{(3)}}{I_{(1)}^3} - \frac{3I_{(0)}^2 I_{(2)} \sigma I_{(2)}}{I_{(1)}^4} + \frac{I_{(0)}^2 + 2W_T I_{(0)} \sigma I_{(1)}}{I_{(1)}^2} - \frac{2W_T I_{(0)}^2 \sigma I_{(2)}}{I_{(1)}^3} \\ = \frac{2\sigma W_T I_{(0)}}{I_{(1)}} - \frac{2\sigma I_{(0)} I_{(2)} + I_{(0)}^2}{I_{(1)}^2} + \frac{\sigma I_{(0)}^2 I_{(3)} - 2\sigma W_T I_{(0)}^2 I_{(2)}}{I_{(1)}^3} - \frac{3\sigma I_{(0)}^2 I_{(2)}^2}{I_{(1)}^4}. \quad (102)$$

$$\Gamma = \frac{e^{-rT}}{x^2 \sigma} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \left(\frac{-\sigma W_T I_{(0)} - 2\sigma W_T I_{(0)}}{I_{(1)}} + \frac{\sigma I_{(0)} I_{(2)} + W_T^2 I_{(0)}^2 + 2\sigma I_{(0)} I_{(2)} - I_{(0)}^2}{I_{(1)}^2} \right. \right. \\ \left. \left. + \frac{W_T I_{(0)}^2 I_{(2)} - \sigma I_{(0)}^2 I_{(3)} + 2\sigma W_T I_{(0)}^2 I_{(2)}}{I_{(1)}^3} + \frac{3\sigma I_{(0)}^2 I_{(2)}^2}{I_{(1)}^4} \right) \right] \\ = \frac{e^{-rT}}{x^2 \sigma} E \left[\Phi \left(\frac{I_{(0)}}{T} \right) \left(\frac{-3\sigma W_T I_{(0)}}{I_{(1)}} + \frac{3\sigma I_{(0)} I_{(2)} + W_T^2 I_{(0)}^2 - I_{(0)}^2}{I_{(1)}^2} \right. \right. \\ \left. \left. + \frac{(2\sigma + 1)W_T I_{(0)}^2 I_{(2)} - \sigma I_{(0)}^2 I_{(3)}}{I_{(1)}^3} + \frac{3\sigma I_{(0)}^2 I_{(2)}^2}{I_{(1)}^4} \right) \right]. \quad (103)$$

□